Proof of P-2

$$x(t) \longleftrightarrow x[k] \qquad y[k] = e^{-jk\frac{2\pi}{T}t_0}\, x[k]$$

$$y(t) \longleftrightarrow x(t-t_0)$$

$$y[k] = \frac{1}{T}\int_T y(t)\, e^{-jk\frac{2\pi}{T}t}\, dt = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\frac{2\pi}{T}t}\, dt \qquad t' = t - t_0 \Rightarrow t = t' + t_0$$

$$= \frac{1}{T}\int_T x(t')\, e^{-jk\frac{2\pi}{T}(t'+t_0)}\, dt = \frac{1}{T}\int_T x(t')\, e^{-jk\frac{2\pi}{T}t_0}\, e^{-jk\frac{2\pi}{T}t'}\, dt$$

$$= e^{-jk\frac{2\pi}{T}t_0} \cdot \underbrace{\frac{1}{T}\int_T x(t')\, e^{-jk\frac{2\pi}{T}t'}\, dt'}_{x[k]} \Rightarrow \boxed{y[k] = e^{-jk\frac{2\pi}{T}t_0} \cdot x[k]}$$

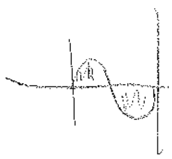
Example

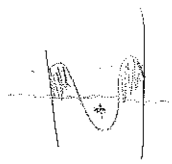
$$\int_T e^{j(k-m)\frac{2\pi}{T}t}\, dt = ?$$

$$\int_T e^{j(k-m)\frac{2\pi}{T}t}\, dt = \int_T \cos\left((k-m)\frac{2\pi}{T}t\right) + j\sin\left((k-m)\frac{2\pi}{T}t\right) dt$$

$$\boxed{\text{if } k \neq m} \Rightarrow \int_T e^{j(k-m)\frac{2\pi}{T}t}\, dt = 0 \qquad \boxed{\text{if } k = m} \Rightarrow \int_T e^{j(k-m)\frac{2\pi}{T}t}\, dt = \int_T e^{j(0)}\, dt = T$$

$$\boxed{\text{Thus, } \int_T e^{j(k-m)\frac{2\pi}{T}t}\, dt = \begin{cases} T & k = m \\ 0 & k \neq m \end{cases}}$$





Proof of Property 6

$$\underbrace{\int_T x(\tau)\, y(t-\tau)\, d\tau}_{\text{periodic convolution}} \xleftrightarrow{FSC} T \cdot x[k] \cdot y[k]$$

$$z(t) = x(t) \circledast y(t) = \int x(\tau)\, y(t-\tau)\, d\tau$$

$$x(\tau) = \sum_k x[k]\, e^{jk\frac{2\pi}{T}\tau}$$

$$y(t) = \sum_m y[m]\, e^{jm\frac{2\pi}{T}t}$$

$$y(t-\tau) = \sum_m y[m]\, e^{jm\frac{2\pi}{T}(t-\tau)}$$

$$z(t) = \int_T \sum_k x[k]\, e^{jk\frac{2\pi}{T}\tau} \sum_m y[m]\, e^{jm\frac{2\pi}{T}(t-\tau)}\, d\tau$$

$$z(t) = \sum_k \sum_m x[k]\, y[m] \int_T e^{j(k-m)\frac{2\pi}{T}\tau}\, d\tau\; e^{jm\frac{2\pi}{T}t}$$

$$= \begin{cases} T & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$