Example

$x[n] = \cos(\Omega_0 n)\, u[n] \qquad X(z) = ?$

$\cos(\Omega_0 n) = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)$

$x[n] = \frac{1}{2}\left[e^{j\Omega_0 n} u(n) + e^{-j\Omega_0 n} u(n)\right]$

$\alpha^n u(n) \xleftrightarrow{z} \dfrac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$

$e^{j\Omega_0 n} u(n) \xleftrightarrow{z} \dfrac{1}{1-e^{j\Omega_0} z^{-1}} \qquad |z| > \underbrace{|e^{j\Omega_0}|}_{1}$

$e^{-j\Omega_0 n} u(n) \xleftrightarrow{z} \dfrac{1}{1-e^{-j\Omega_0} z^{-1}} \qquad |z| > 1$

Thus,

$X(z) = \frac{1}{2}\left[ZT\{e^{j\Omega_0 n} u[n]\} + ZT\{e^{-j\Omega_0 n} u[n]\}\right]$

$= \frac{1}{2}\left[\dfrac{1}{1-e^{j\Omega_0} z^{-1}} + \dfrac{1}{1-e^{-j\Omega_0} z^{-1}}\right]$

$\boxed{= \frac{1}{2}\left[\dfrac{1-e^{-j\Omega_0} z^{-1} + 1 - e^{j\Omega_0} z^{-1}}{(1-e^{-j\Omega_0} z^{-1})(1-e^{j\Omega_0} z^{-1})}\right]}$

Example

$x[n] = \delta(n-m), \quad X(z) = ?$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n-m) z^{-n} = \sum_{n=-\infty}^{\infty} z^{-m} = \boxed{z^{-m}}$

The Inverse z-Transform

z-transform of $x[n]$

$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z = re^{j\Omega}$

$X[z] = FT\{x[n] r^{-n}\} \Rightarrow x[n] r^{-n} = \frac{1}{2\pi}\int_{2\pi} X(z) e^{j\Omega n} d\Omega$

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(z) (re^{j\Omega})^n d\Omega$

$z = re^{j\Omega} \longrightarrow dz = jre^{j\Omega} d\Omega$

$d\Omega = \frac{1}{j} z^{-1} dz$

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(z) z^{n-1} \frac{1}{j} dz$

$= \frac{1}{j2\pi}\oint X(z) z^{n-1} dz$

$\Rightarrow \boxed{x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz}$

Since it is not so easy to calculate this integral everytime, generally known z transform pairs are used instead.

Example

$X(z) = \dfrac{3 - \frac{5}{16} z^{-1}}{\left(1-\frac{1}{4} z^{-1}\right)\left(1-\frac{1}{3} z^{-1}\right)} \qquad |z| > \frac{1}{3}$

$x[n] = ?$

$X(z) = \dfrac{3 - \frac{5}{16} z^{-1}}{\left(1-\frac{1}{4} z^{-1}\right)\left(1-\frac{1}{3} z^{-1}\right)} = \dfrac{A}{\left(1-\frac{1}{4} z^{-1}\right)} + \dfrac{B}{\left(1-\frac{1}{3} z^{-1}\right)}$